

Non blow-up criterion for the 3-D Magneto-hydrodynamics equations in the limiting case

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Abstract

In this paper, we prove that suitable weak solution (u, b) of the 3-D MHD equations can be extended beyond T if $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ and the horizontal components b_h of the magnetic field satisfies the well-known Ladyzhenskaya-Prodi-Serrin condition.

1 Introduction

We consider the 3-D incompressible Magneto-hydrodynamics (MHD) equations as follows:

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u = -\nabla \pi + b \cdot \nabla b, \\ b_t - \Delta b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \quad (1.1)$$

Here u , b describe the fluid velocity field and the magnetic field respectively, p is a scalar pressure. The global existence of weak solution and local existence of strong solution to the MHD equations (1.1) were proved by Duvaut and Lions [6]. As the incompressible Navier-Stokes equations, the regularity and uniqueness of weak solutions remains a challenging open problem. We refer to [15] for some mathematical questions related to the MHD equations.

It is well-known that if the weak solution of the Navier-Stokes equations satisfies the Ladyzhenskaya-Prodi-Serrin condition

$$u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,$$

then it is regular on $(0, T) \times \mathbb{R}^3$. Note that the limiting case $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ does not fall into the framework of energy method, which was proved by Escauriaza-Seregin-Šverák [7]. Wu [18, 19] extended Ladyzhenskaya-Prodi-Serrin type criterions to the MHD equations in terms of both the velocity field u and the magnetic field b for $p > 3$. However, some numerical experiments seem to indicate that the velocity field should play a more important role than the magnetic field in the regularity theory of solutions to the MHD equations [12]. Recently, He-Xin [8] and Zhou [20] have presented some regularity criterions to the MHD

equations in terms of the velocity field only. Chen-Miao-Zhang [3, 4] extend and improve their results as follows: if the weak solution of the MHD equations (1.1) satisfies

$$u \in L^q(0, T; B_{p, \infty}^s) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 + s, \quad \frac{3}{1+s} < p \leq \infty, \quad -1 < s \leq 1,$$

then it is regular on $(0, T) \times \mathbb{R}^3$. Here $B_{p, \infty}^s$ is the Besov space. We refer to [2, 10, 16] and references therein for more relevant results. However, whether the condition on b can be removed remains unknown in the limiting case (i.e., $(p, q, s) = (3, \infty, 0)$). The case $u, b \in L^\infty(0, T; L^3(\mathbb{R}^3))$ was considered by Mahalov-Nicolaenko-Shikin [11], and Wang-Zhang [17] proved that if $u \in L^\infty(-1, 0; L^3(\mathbb{R}^3))$, and

$$b \in L^\infty(-1, 0; BMO^{-1}(\mathbb{R}^3)) \quad \text{and} \quad b(t) \in VMO^{-1}(\mathbb{R}^3) \quad \text{for } t \in (-1, 0].$$

Then (u, b) is Hölder continuous on $\mathbb{R}^3 \times (-1, 0]$.

Note that the inclusion relation: $L^3(\mathbb{R}^3) \subsetneq VMO^{-1}(\mathbb{R}^3)$. It's interesting to ask: whether the condition on the magnetic field can be removed.

Our main result is the following:

Theorem 1.1 *Let (u, b) be a smooth solution of the MHD equations (1.1) in $(-1, 0) \times \mathbb{R}^3$, which is also suitable. Assume that $u \in L^\infty(-1, 0; L^3(\mathbb{R}^3))$ and b satisfies one of the following conditions*

$$\begin{aligned} i) \nabla b_h &\in L_t^q L_x^p((-\frac{1}{2}, 0) \times \mathbb{R}^3), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{9}{4} \leq p < 3; \\ ii) \nabla b_h &\in L_t^q L_x^p((-\frac{1}{2}, 0) \times \mathbb{R}^3), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 3 \leq p \leq \infty, \quad \text{and} \\ b_h &\in L_t^s L_x^l((-\frac{1}{2}, 0) \times \mathbb{R}^3), \quad \frac{3}{l} + \frac{2}{s} = 1, \quad 9 \leq l \leq \infty. \end{aligned} \tag{1.2}$$

Then (u, b) is regular on $\mathbb{R}^3 \times (-1, 0]$.

Remark 1.2 *For simplicity, we assume that (u, b) is smooth and suitable. At this moment, one can integrate by parts legitimately and the energy norms are finite, i.e.*

$$\|u\|_{L^\infty(-1, 0; L^2(\mathbb{R}^3)) \cap L^2(-1, 0; \dot{H}^1(\mathbb{R}^3))} + \|b\|_{L^\infty(-1, 0; L^2(\mathbb{R}^3)) \cap L^2(-1, 0; \dot{H}^1(\mathbb{R}^3))} < \infty. \tag{1.3}$$

Motivated by the theory of Escauriza-Seregin-Šverák [7], the main difficulty lies in proving that some necessary scaling invariant quantities of u, b are bounded for compactness arguments. Using (1.2), the vertical component b_3 of the magnetic field can be estimated by the energy method technically. On the other hand, the key is a class of careful interior regular criteria. In fact, let

$$G(b_h, p, q; r) \equiv r^{1-\frac{3}{p}-\frac{2}{q}} \|b_h\|_{L_t^q L_x^p(Q_r)},$$

where $Q_r = (-r^2, 0) \times B_r$ and B_r is a ball of radius r centered at zero. We have the following more general interior criteria in the limiting case:

Theorem 1.3 *Let (u, b) be a suitable weak solution of the MHD equations (1.1) in $(-1, 0) \times B_1$. Assume that $u \in L^\infty(-1, 0; L^3(B_1))$, and b satisfies the following conditions*

$$\begin{aligned} i) \liminf_{r \rightarrow 0} G(b_h, p, q; r) &= 0, \quad \sup_{0 < r < 1} G(b_h, p, q; r) < \infty, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq p \leq \infty; \\ ii) \sup_{0 < r < 1} G(b_3, l, s; r) &< \infty, \quad \frac{3}{l} + \frac{2}{s} = 2, \quad 1 \leq s \leq \infty. \end{aligned}$$

Then (u, b) is regular on $\mathbb{R}^3 \times (-1, 0]$.

For $u \in L_t^\infty L_x^3$, whether $b \in L_t^\infty(BMO_x^{-1})$ or the above condition of b without

$$\liminf_{r \rightarrow 0} G(b_h, p, q; r) = 0$$

implies the regularity of (u, b) is still unknown, where standard energy methods or backward uniqueness methods seem to be out of reach.

2 Preliminaries

Let us first introduce the definitions of suitable weak solution.

Definition 2.1 *Let $T > 0$ and $\Omega \subset \mathbb{R}^3$. We say that (u, b) is a suitable weak solution of the MHD equations (1.1) in $\Omega_T = \Omega \times (-T, 0)$ if*

- (a) $(u, b) \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H_0^1(\Omega))$;
- (b) (u, b, π) satisfies the equations (1.1) in $\mathcal{D}'(\Omega_T)$;
- (c) $\pi \in L^{\frac{3}{2}}(\Omega_T)$ and the following local energy inequality holds: for a.e. $t \in [-T, 0]$

$$\begin{aligned} & \int_{\Omega} (|u(x, t)|^2 + |b(x, t)|^2) \phi dx + 2 \int_{-T}^t \int_{\Omega} (|\nabla u|^2 + |\nabla b|^2) \phi dx ds \\ & \leq \int_{-T}^t \int_{\Omega} [(|u|^2 + |b|^2)(\Delta \phi + \partial_s \phi) + u \cdot \nabla \phi (|u|^2 + |b|^2 + 2\pi) - (b \cdot u)(b \cdot \nabla \phi)] dx ds, \end{aligned}$$

for any nonnegative $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ vanishing in a neighborhood of the parabolic boundary of Ω_T .

We define a solution (u, b) to be regular at $z_0 = (x_0, t_0)$ if $(u, b) \in L^\infty(Q_r(z_0))$ with $Q_r(z_0) = (-r^2 + t_0, t_0) \times B_r(x_0)$, and $B_r(x_0)$ is a ball of radius r centered at x_0 . We also denote Q_r by $Q_r(0)$ and B_r by $B_r(0)$. For a function u defined on $Q_r(z_0)$, the mixed space-time norm $\|u\|_{L^{p,q}(Q_r(z_0))}$ is defined by

$$\|u\|_{L^{p,q}(Q_r(z_0))}^q := \int_{t_0-r^2}^{t_0} \left(\int_{B_r(x_0)} |u(x, t)|^p dx \right)^{\frac{q}{p}} dt.$$

The following small energy regularity result is well-known, see [8, 11].

Proposition 2.2 *Assume that (u, b) is a suitable weak solution of (1.1) in $Q_1(z_0)$. There exists an absolute constant $\varepsilon > 0$ such that if*

$$r^{-2} \int_{Q_r(z_0)} |u|^3 + |b|^3 + |\pi|^{\frac{3}{2}} dx dt \leq \varepsilon$$

for some $r > 0$, then (u, b) is regular at the point z_0 .

We also need the small energy interior regularity result in terms of the velocity only in [16], and the according boundary regularity result see [?].

Proposition 2.3 *Assume that (u, b) is a suitable weak solution of (1.1) in $Q_1(z_0)$. There exists an absolute constant $\varepsilon > 0$ such that if $u \in L^{p,q}$ near z_0 and*

$$\limsup_{r \rightarrow 0+} r^{-(\frac{3}{p} + \frac{2}{q} - 1)} \|u\|_{L^{p,q}(Q_r(z_0))} < \varepsilon, \quad (2.1)$$

with p, q satisfying $1 \leq \frac{3}{p} + \frac{2}{q} \leq 2$, $1 \leq q \leq \infty$ and $(p, q) \neq (\infty, 1)$. Then (u, b) is regular at the point z_0 .

Let (u, b, π) be a solution of (1.1) and introduce the following scaling:

$$u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad b^\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t), \quad \pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t), \quad (2.2)$$

for any $\lambda > 0$, then the family $(u^\lambda, b^\lambda, \pi^\lambda)$ is also a solution of (1.1). For $z_0 = (x_0, t_0)$, we define some invariant quantities under the scaling (2.2):

$$\begin{aligned} A(f, r) &= \sup_{-r^2 \leq t < 0} r^{-1} \int_{B_r} |f(y, t)|^2 dy, & C(f, r) &= r^{-2} \int_{Q_r} |f(y, s)|^3 dy ds, \\ E(f, r) &= r^{-1} \int_{Q_r} |\nabla f(y, s)|^2 dy ds, & K(f, r) &= r^{-3} \int_{Q_r} |f(y, s)|^2 dy ds, \end{aligned}$$

for $f = u, b$ and

$$\begin{aligned} D(\pi, r) &= r^{-2} \int_{Q_r} |\pi(y, s)|^{\frac{3}{2}} dy ds, \\ \tilde{D}(\pi, r) &= r^{-2} \int_{Q_r} |\pi(y, s) - (\pi)_{B_r}|^{\frac{3}{2}} dy ds, & (\pi)_{B_r} &= \frac{1}{|B_r|} \int_{B_r} \pi(y, s) dy. \end{aligned}$$

Let $A(u, b; r) = A(u, r) + A(b, r)$, and $E(u, b; r)$, $C(u, b; r)$ and $K(u, b; r)$ denote similar notations. We also introduce

$$\begin{aligned} G(f, p, q; r) &= r^{1 - \frac{3}{p} - \frac{2}{q}} \|f\|_{L^{p,q}(Q_r)}, & \tilde{G}(f, p, q; r) &= r^{1 - \frac{3}{p} - \frac{2}{q}} \|f - (f)_{B_r}\|_{L^{p,q}(Q_r)}, \\ H(f, p, q; r) &= r^{2 - \frac{3}{p} - \frac{2}{q}} \|f\|_{L^{p,q}(Q_r)}, & \tilde{H}(f, p, q; r) &= r^{2 - \frac{3}{p} - \frac{2}{q}} \|f - (f)_{B_r}\|_{L^{p,q}(Q_r)}. \end{aligned}$$

Throughout this paper, we denote by C_0 a constant independent of r, ρ and different from line to line.

3 Proof of Theorem 1.1

In this section, we'll prove Theorem 1.1 with the help of Theorem 1.3. Moreover, we assume that

$$\|u\|_{L_t^\infty L_x^3((-1,0) \times R^3)} + \|b(\cdot, -\frac{1}{2})\|_{L_x^3(R^3)} \leq C_0,$$

which is reasonable from the assumptions of Theorem 1.1 and (1.3).

First, we have the following embedding inequality and Sobolev's interpolation inequality (for example, see [1]):

Lemma 3.1 *i) For $2 \leq \ell \leq 6$, $a = \frac{3}{4}(\ell - 2)$ and $f \in H^1(R^3)$, we have*

$$\int_{R^3} |f|^\ell \leq C_0 \left(\int_{R^3} |\nabla f|^2 \right)^a \left(\int_{R^3} |f|^2 \right)^{\frac{\ell}{2}-a}. \quad (3.1)$$

ii) For $f \in L^\infty(-1, 0; L^2(R^3)) \cap L^2(-1, 0; \dot{H}^1(R^3))$, we have

$$\|f\|_{L_t^s L_x^l((-1,0) \times R^3)} \leq C_0 \|f\|_{L_t^\infty L_x^2((-1,0) \times R^3)}^{1-\frac{2}{s}} \|f\|_{L_t^2 \dot{H}_x^1((-1,0) \times R^3)}^{\frac{2}{s}}, \quad (3.2)$$

where $\frac{3}{l} + \frac{2}{s} = \frac{3}{2}$ with $2 \leq s \leq \infty$.

Lemma 3.2 *Under the assumption of Theorem 1.1, we have*

$$|b_3|^{\frac{3}{2}} \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1\left(-\frac{1}{2}, 0\right) \times \mathbb{R}^3).$$

Proof. Recall the third equation of the magnetic field:

$$\partial_t b_3 - \Delta b_3 + u \cdot \nabla b_3 = b \cdot \nabla u_3,$$

and multiplying $3|b_3|b_3$ on both sides of it, we have

$$\partial_t (|b_3|^3) - 3\Delta b_3 (|b_3|b_3) + u \cdot \nabla (|b_3|^3) = 3b \cdot \nabla u_3 (|b_3|b_3).$$

Integrating on R^3 , using integration by parts and $\nabla \cdot u = 0$ we derive that

$$\partial_t \int_{R^3} |b_3|^3 dx + \frac{8}{3} \int_{R^3} |\nabla (|b_3|^{\frac{3}{2}})|^2 dx = 3 \int_{R^3} (b \cdot \nabla u_3) (|b_3|b_3) dx \equiv 3I. \quad (3.3)$$

Let $\nabla_h = (\partial_1, \partial_2)^T$ and $I = I_1 + I_2$, where

$$I_1 = \int_{R^3} (b_h \cdot \nabla_h u_3) (|b_3|b_3) dx, \quad I_2 = \int_{R^3} (b_3 \partial_3 u_3) (|b_3|b_3) dx.$$

Obviously,

$$\begin{aligned} I_1 &\leq \left| \int_{R^3} (\nabla_h \cdot b_h) u_3 (|b_3|b_3) dx \right| + 2 \int_{R^3} |b_h| |u_3| |b_3| |\nabla_h b_3| dx \\ &\leq C_0 \|u_3\|_{L^3(R^3)} [\|\nabla_h \cdot b_h |b_3|^2\|_{L^{\frac{3}{2}}(R^3)} + \|b_3 b_h |\nabla_h b_3|\|_{L^{\frac{3}{2}}(R^3)}] \\ &\leq C_0 [\|\nabla_h \cdot b_h |b_3|^2\|_{L^{\frac{3}{2}}(R^3)} + \|b_3 b_h |\nabla_h b_3|\|_{L^{\frac{3}{2}}(R^3)}], \end{aligned}$$

and the divergence-free property of b implies that

$$\begin{aligned} I_2 &\leq C_0 \int_{R^3} |u_3| |\partial_3 b_3| |b_3|^2 dx \\ &\leq C_0 \int_{R^3} |u_3| |\nabla_h \cdot b_h |b_3|^2| dx \leq C_0 \|\nabla_h \cdot b_h |b_3|^2\|_{L^{\frac{3}{2}}(R^3)}. \end{aligned}$$

From the above estimates of I_1, I_2 and (3.3), we derive that

$$\partial_t \int_{R^3} |b_3|^3 dx + \frac{8}{3} \int_{R^3} |\nabla (|b_3|^{\frac{3}{2}})|^2 dx \leq C_0 [\|\nabla_h \cdot b_h |b_3|^2\|_{L^{\frac{3}{2}}(R^3)} + \|b_3 b_h |\nabla_h b_3|\|_{L^{\frac{3}{2}}(R^3)}], \quad (3.4)$$

and let

$$II_1 \equiv \|\nabla_h \cdot b_h |b_3^2|\|_{L^{\frac{3}{2}}(R^3)}, \quad II_2 \equiv \|b_3 b_h |\nabla_h b_3|\|_{L^{\frac{3}{2}}(R^3)}.$$

Step I: Estimate of II_1 . Let $\frac{3}{p} + \frac{2}{q} = 2$ with $1 \leq q \leq \infty$ and

$$\|\nabla b_h\|_{L_t^q L_x^p((-\frac{1}{2}, 0) \times R^3)} \leq C_0.$$

Then by Hölder inequality we have

$$II_1 \leq \|\nabla b_h\|_{L_x^p(R^3)} \|b_3\|_{L_x^{\frac{6p}{2p-3}}(R^3)}^2,$$

where the end case $p = \infty$ or $p = \frac{3}{2}$ still holds for the above inequality. For any τ with $-\frac{1}{2} < \tau < 0$, integrating to time we have

$$\int_{-\frac{1}{2}}^{\tau} II_1 dt \leq \|\nabla b_h\|_{L_t^q L_x^p((-\frac{1}{2}, \tau) \times R^3)} \|b_3\|_{L_t^{2q'} L_x^{\frac{6p}{2p-3}}((-\frac{1}{2}, \tau) \times R^3)}^2,$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

To apply Gronwall's inequality, we choose $q < \infty$ and by Lemma 3.1 q', p should satisfy

$$\frac{2p-3}{2p} + \frac{1}{q'} = 1, \quad 3 \leq 2q' \leq \infty,$$

which yields that $1 \leq q \leq 3$ or $\frac{9}{4} \leq p \leq \infty$. Hence

$$\|b_3\|_{L_t^{2q'} L_x^{\frac{6p}{2p-3}}((-\frac{1}{2}, \tau) \times R^3)} \leq C_0 \left[\|(|b_3|^{\frac{3}{2}})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times R^3)} + \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, \tau) \times R^3)} \right]^{\frac{2}{3}}. \quad (3.5)$$

Step II: Estimate of II_2 .

Let $\frac{3}{l} + \frac{2}{s} = 1$ with $2 \leq s \leq \infty$ and

$$\|b_h\|_{L_t^s L_x^l((-\frac{1}{2}, 0) \times R^3)} \leq C_0.$$

Then by Hölder inequality we have

$$II_2 \leq C_0 \|b_h\|_{L_x^l(R^3)} \|\nabla(|b_3|^{\frac{3}{2}})\|_{L^2(\mathbb{R}^3)} \|b_3\|_{L_x^{\frac{3p_1}{4}}(R^3)}^{\frac{1}{2}},$$

where p_1 satisfies

$$\frac{3}{2l} + \frac{3}{4} + \frac{1}{p_1} = 1, \quad 1 \leq p_1 \leq \infty.$$

Integrating to time with the same τ as above, we have

$$\int_{-\frac{1}{2}}^{\tau} II_2 dt \leq \|b_h\|_{L_t^s L_x^l((-\frac{1}{2}, \tau) \times R^3)} \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, \tau) \times R^3)} \|b_3\|_{L_t^{\frac{q_1}{2}} L_x^{\frac{3p_1}{4}}((-\frac{1}{2}, \tau) \times R^3)}^{\frac{1}{2}},$$

where q_1 satisfies

$$\frac{1}{s} + \frac{1}{2} + \frac{1}{q_1} = 1, \quad 1 \leq q_1 \leq \infty.$$

Using $\frac{3}{l} + \frac{2}{s} = 1$, we have

$$\frac{4}{p_1} + \frac{4}{q_1} = 1,$$

that is $|b_3|^{\frac{3}{2}} \in L_t^{\frac{q_1}{3}} L_x^{\frac{p_1}{2}}$, and

$$3\frac{2}{p_1} + 2\frac{3}{q_1} = \frac{3}{2},$$

which yields that by Lemma 3.1, for $2 \leq \frac{q_1}{3} \leq \infty$,

$$\|b_3\|_{L_t^{\frac{q_1}{2}} L_x^{\frac{3p_1}{4}}((-\frac{1}{2}, \tau) \times R^3)} \leq C_0 [\|(|b_3|^{\frac{3}{2}})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times R^3)} + \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, \tau) \times R^3)}]^{\frac{2}{3}}. \quad (3.6)$$

Then s must be $2 \leq s \leq 3$ or $9 \leq l \leq \infty$.

Step III: Arguments. From the above two estimates, we obtain that if the condition $i)$ of (1.2) holds, i.e.

$$\|\nabla b_h\|_{L_t^q L_x^p((-\frac{1}{2}, 0) \times R^3)} \leq C_0, \quad \frac{9}{4} \leq p < 3,$$

then the embedding inequality implies

$$\|b_h\|_{L_t^q L_x^l((-\frac{1}{2}, 0) \times R^3)} \leq C_0, \quad \frac{3}{l} + \frac{2}{q} = 1, \quad 9 \leq l < \infty,$$

thus the estimates (3.5)-(3.6) hold, which yields that for $-\frac{1}{2} < \tau < 0$, there holds

$$\begin{aligned} & \int_{-\frac{1}{2}}^{\tau} \int_{R^3} \partial_t(|b_3|^3) dx dt + \frac{8}{3} \int_{-\frac{1}{2}}^{\tau} \int_{R^3} |\nabla(|b_3|^{\frac{3}{2}})|^2 dx dt \\ & \leq C_0 [\|(|b_3|^{\frac{3}{2}})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times R^3)} + \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, \tau) \times R^3)}]^{\frac{4}{3}}. \end{aligned} \quad (3.7)$$

Note that $\|b(\cdot, -\frac{1}{2})\|_{L_x^3(R^3)} \leq C_0$. Hence, Gronwall's inequality implies

$$\|(|b_3|^{\frac{3}{2}})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times R^3)} + \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, \tau) \times R^3)} < \infty,$$

and taking the supremum of $\tau \rightarrow 0$, we have

$$\|(|b_3|^{\frac{3}{2}})\|_{L_t^\infty L_x^2((-\frac{1}{2}, 0) \times R^3)} + \|\nabla(|b_3|^{\frac{3}{2}})\|_{L_{t,x}^2((-\frac{1}{2}, 0) \times R^3)} \leq C_0. \quad (3.8)$$

On the other hand, when the condition $ii)$ of (1.2) holds, simialr estimates hold. The lemma is proved. \square

Proof of Theorem 1.1: Due to the assumptions on b_h , we have

$$\|b_h\|_{L_t^s L_x^l((-\frac{1}{2}, 0) \times R^3)} \leq C_0, \quad \frac{3}{l} + \frac{2}{s} = 1, \quad 9 \leq l \leq \infty.$$

Moreover, Lemma 3.2 yields that

$$\|b_3\|_{L_{t,x}^5((-\frac{1}{2}, 0) \times R^3)} < \infty.$$

Obviously, the conditions of Theorem 1.3 are satisfied, thus the proof is complete. \square

4 Blow-up analysis and Proof of Theorem 1.3

We will apply Proposition 2.2 to prove the interior regularity of the solution by blow-up analysis, which was early used for the 3D Navier-Stokes equations in [?], see also [14]. Note that the velocity u is in the critical class, hence backward uniqueness results in [7] are still needed. Firstly, we prove that the basic energy norms $A(u, b; r)$, $E(u, b; r)$, and $\tilde{D}(\pi, r)$ are uniformly bounded for all $0 < r < 1$. (see Theorem 4.1); secondly, a standard compactness argument for suitable weak solutions of the 3-D MHD equations and backward uniqueness results imply Theorem 1.3.

4.1 Bounded estimates of $A(u, b; r)$ and $E(u, b; r)$

To ensure the validness of blow-up analysis, we have to prove that $A(u, b; r)$, $E(u, b; r)$, and $\tilde{D}(\pi, r)$ are uniformly bounded for all $0 < r < r_1$ with some $r_1 > 0$.

Theorem 4.1 *Under the assumptions of 1.3, there exists a $r_1 > 0$ such that*

$$A(u, b; r) + E(u, b; r) + \tilde{D}(\pi, r) < \infty, \quad 0 < r < r_1, \quad (4.1)$$

where r_1 depends on $C(u, b; 1)$ and $\tilde{D}(\pi, 1)$.

For completeness, we supply the following technical lemmas. First, we will control $A(u, b; r) + E(u, b; r)$ in terms of the other scaling invariant quantities by using the following local inequality.

Lemma 4.2 *Let $0 < 4r < \rho < r_0$ and $1 \leq p, q \leq \infty$. There holds*

$$\begin{aligned} & A(u, b; r) + E(u, b; r) \\ & \leq C_0 \left(\frac{r}{\rho}\right)^2 K(u, b; \rho) + C_0 \left(\frac{\rho}{r}\right)^2 \left[C(u, \rho) + C(u, \rho)^{1/3} (C(b, \rho)^{2/3} + \tilde{D}(\pi, \rho)^{2/3}) \right]. \end{aligned}$$

Proof. Let ζ be a cutoff function, which vanishes outside of Q_ρ and equals 1 in $Q_{\rho/2}$, and satisfies

$$|\nabla \zeta| \leq C_0 \rho^{-1}, \quad |\partial_t \zeta|, |\Delta \zeta| \leq C_0 \rho^{-2}.$$

Define the backward heat kernel as

$$\Gamma(x, t) = \frac{1}{4\pi(r^2 - t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4(r^2 - t)}}.$$

Taking the test function $\phi = \Gamma\zeta$ in the local energy inequality, and noting $(\partial_t + \Delta)\Gamma = 0$, we have

$$\begin{aligned} & \sup_t \int_{B_\rho} (|u|^2 + |b|^2) \phi dx + \int_{Q_\rho} (|\nabla u|^2 + |\nabla b|^2) \phi dx dt \\ & \leq \int_{Q_\rho} [(|u|^2 + |b|^2)(\Gamma\Delta\zeta + \Gamma\partial_t\zeta + 2\nabla\Gamma \cdot \nabla\zeta) + |\nabla\phi||u|(|u|^2 + |b|^2 + 2|\pi - \pi_{B_\rho}|)] dx dt. \end{aligned}$$

Some direct computations imply

$$\begin{aligned} \Gamma(x, t) & \geq C_0^{-1} r^{-3} \quad \text{in } Q_r; \\ |\nabla\phi| & \leq |\nabla\Gamma|\zeta + \Gamma|\nabla\zeta| \leq C_0 r^{-4}; \\ |\Gamma\Delta\zeta| + |\Gamma\partial_t\zeta| + 2|\nabla\Gamma \cdot \nabla\zeta| & \leq C_0 \rho^{-5}, \end{aligned}$$

from which and Hölder inequality, Lemma 4.2 follows. \square

The following is an interpolation inequality.

Lemma 4.3 *For any $0 < r < r_0$, let $\frac{3}{p} + \frac{2}{q} = 2$ with $1 \leq q \leq \infty$, there holds*

$$C(f, r) \leq C_0 G(f, p, q; r) (E(f, r) + A(f, r)),$$

where $f = u, b$.

Proof. Without loss of generality, we consider the estimate of u . By Hölder inequality and Sobolev inequality, we get

$$\begin{aligned} \int_{B_r} |u|^3 dx &= \int_{B_r} |u|^{3\alpha+3\beta+3-3\alpha-3\beta} dx \\ &\leq \left(\int_{B_r} |u|^2 dx \right)^{3\alpha/2} \left(\int_{B_r} |u|^6 dx \right)^{\beta/2} \left(\int_{B_r} |u|^p dx \right)^{(3-3\alpha-3\beta)/p} \\ &\leq C_0 \left(\int_{B_r} |u|^2 dx \right)^{3\alpha/2} \left(\int_{B_r} |\nabla u|^2 + |u|^2 dx \right)^{3\beta/2} \left(\int_{B_r} |u|^p dx \right)^{(3-3\alpha-3\beta)/p}, \end{aligned}$$

where α, β are chosen so that

$$\frac{1}{3} = \frac{\alpha}{2} + \frac{\beta}{6} + \frac{1-\alpha-\beta}{p}, \quad \frac{3\beta}{2} + \frac{3-3\alpha-3\beta}{q} = 1.$$

Taking $\alpha = \frac{2p-3}{3p}$ and $\beta = \frac{1}{p}$, we get

$$\begin{aligned} \int_{Q_r} |u|^3 dx &\leq C_0 \left(\sup_{-r^2 < t < 0} \int_{B_r} |u|^2 dx \right)^{1-\frac{3}{2p}} \left(\int_{Q_r} |\nabla u|^2 + |u|^2 dx dt \right)^{\frac{3}{2p}} \\ &\quad \times \left(\int_{-r^2}^0 \left(\int_{B_r} |u|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}, \end{aligned}$$

which yields the required inequality. \square

We present the estimate of the pressure in terms of scaling invariant quantities, see also [13].

Lemma 4.4 *Let (u, b) be a suitable weak solution of (1.1) in Q_1 . Then there hold*

$$\tilde{D}(\pi, r) \leq C_0 \left(\left(\frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi, \rho) + \left(\frac{\rho}{r} \right)^2 C(u, b; \rho) \right), \quad (4.2)$$

for any $0 < 4r < \rho < 1$.

Proof. Note that π satisfies the following equation in distribution sense:

$$-\Delta \pi = \partial_i \partial_j (\hat{u}_i \hat{u}_j - \hat{b}_i \hat{b}_j),$$

where $\hat{u} = u - (u)_{B_\rho}$ and $\hat{b} = b - (b)_{B_\rho}$. Let ζ be a cut-off function, which equals 1 in $Q_{\rho/2}$ and vanishes outside of Q_ρ . Set $\pi = \pi_1 + \pi_2$ with

$$\pi_1 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} [\partial_i \partial_j ((\hat{u}_i \hat{u}_j - \hat{b}_i \hat{b}_j) \zeta^2)],$$

and π_2 is harmonic in $Q_{\rho/2}$.

Due to the Calderon-Zygmund inequality, we have

$$\int_{B_\rho} |\pi_1|^{\frac{3}{2}} dx \leq C_0 \int_{B_\rho} |\hat{u}|^3 + |\hat{b}|^3 dx.$$

Since π_2 is harmonic in $Q_{\rho/2}$, we have

$$\begin{aligned} \int_{B_r} |\pi_2 - (\pi_2)_{B_r}|^{\frac{3}{2}} dx &\leq C_0 r^{3+\frac{3}{2}} \sup_{B_{\rho/4}} |\nabla \pi_2|^{\frac{3}{2}} \\ &\leq C_0 \left(\frac{r}{\rho} \right)^{3+\frac{3}{2}} \int_{B_{\rho/2}} |\pi_2 - (\pi_2)_{B_{\rho/2}}|^{\frac{3}{2}} dx. \end{aligned}$$

Hence we infer that

$$\tilde{D}(\pi, r) \leq \tilde{D}(\pi_1, r) + \tilde{D}(\pi_2, r) \leq C_0 \left(\left(\frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi, \rho) + \left(\frac{\rho}{r} \right)^2 C(u, b; \rho) \right).$$

□

Proof of Theorem 4.1. Without loss of generality, we assume that

$$\sup_{0 < r < 1} G(b_h, p, q; r) + \sup_{0 < r < 1} G(b_3, l, s; r) \leq C_0,$$

for some (p, q) and (l, s) satisfying

$$\frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{l} + \frac{2}{s} = 2.$$

Then, by Lemma 4.3 we have

$$C(b, \rho) \leq C_0 [C(b_h, \rho) + C(b_3, \rho)] \leq C_0 [A(b, \rho) + E(b, \rho)].$$

Thus, by the local energy inequality and $C(u, \rho) \leq C_0$, we have

$$\begin{aligned} & A(u, b; r) + E(u, b; r) \\ & \leq C_0 \left(\frac{r}{\rho}\right)^2 A(u, b; \rho) + C_0 \left(\frac{\rho}{r}\right)^2 \left[1 + (C(b, \rho)^{2/3} + \tilde{D}(\pi, \rho)^{2/3})\right] \\ & \leq C_0 \left(\frac{r}{\rho}\right)^2 [A(u, b; \rho) + E(u, b; \rho)] + C_0 \left(\frac{\rho}{r}\right)^2 \left[\left(\frac{\rho}{r}\right)^8 + \tilde{D}(\pi, \rho)^{2/3}\right]. \end{aligned}$$

Take $0 < 8r < \rho < r_0$ and set

$$F(r) = A(u, b; r) + E(u, b; r) + \varepsilon^{-1/2} \tilde{D}(\pi, r)^{2/3}.$$

Due to Lemma 4.4, we get

$$\begin{aligned} \tilde{D}(\pi, r) & \leq C_0 \left(\left(\frac{r}{\rho}\right)^{5/2} \tilde{D}(\pi, \rho) + \left(\frac{\rho}{r}\right)^2 C(u, b; \rho)\right) \\ & \leq C_0 \left(\frac{r}{\rho}\right)^{5/2} \tilde{D}(\pi, \rho) + C_0 \left(\frac{\rho}{r}\right)^2 (1 + A(b, \rho) + E(b, \rho)) \end{aligned}$$

Hence,

$$F(r) \leq C_0 \left[\left(\frac{r}{\rho}\right)^2 + \left(\frac{\rho}{r}\right)^2 \varepsilon^{1/2} + \left(\frac{r}{\rho}\right)^{5/3}\right] F(\rho) + C_0 \varepsilon^{-3/2} \left(\frac{\rho}{r}\right)^{10},$$

and choosing $\theta = \left(\frac{\rho}{r}\right)$ and ε sufficiently small, we get

$$F(\theta\rho) \leq \frac{1}{2} F(\rho) + C_0 \varepsilon^{-3/2} \theta^{-10},$$

which yields that there exists a $r_1 > 0$ such that

$$\sup_{0 < r < r_1} F(r) < C_1,$$

where C_1 depends on C_0 , $C(u, b; 1)$, $\tilde{D}(\pi, 1)$. □

4.2 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the blow-up analysis and unique continuation theorem, for example see [14], [7].

We assume that $\|u\|_{L_t^\infty L_x^3(Q_1)} \leq C_0$, and

$$\begin{aligned} i) \quad & \liminf_{r \rightarrow 0} G(b_h, p, q; r) = 0, \quad \limsup_{0 < r < 1} G(b_h, p, q; r) \leq C_0, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq p \leq \infty; \\ ii) \quad & \sup_{0 < r < 1} G(b_3, l, s; r) < C_0, \quad \frac{3}{l} + \frac{2}{s} = 2, \quad 1 \leq s \leq \infty. \end{aligned}$$

Then, by the local energy inequality in Proposition ?? and Theorem 4.1, we have

$$A(u, b; r) + E(u, b; r) + \tilde{D}(\pi, r) \leq C_1, \quad \text{for all } 0 < r < 1,$$

where $C_1 > 0$ may depend on $C(u, b; 1)$ and $D(\pi, 1)$. Moreover, suppose that $C(u, b; 1) + D(\pi, 1) \leq C_0$, which is reasonable by the definition of suitable weak solutions.

Suppose that the statement of the theorem is false. Then there exist a series of suitable weak solutions $(v^k, \bar{b}^k, \bar{\pi}^k)$ and $r_k \downarrow 0$ such that

$$A(v^k, \bar{b}^k; r) + E(v^k, \bar{b}^k; r) + \tilde{D}(\bar{\pi}^k, r) \leq C_1, \quad \text{for all } 0 < r < 1, \quad (4.3)$$

and

$$G(\bar{b}_h^k, p, q; r_k) \rightarrow 0, \quad \text{as } r_k \rightarrow 0.$$

Moreover, $(0, 0)$ is a singular point of $(v^k, \bar{b}^k, \bar{\pi}^k)$.

We denote

$$u^k(y, s) = r_k v^k(r_k y, r_k^2 s), \quad b^k(y, s) = r_k \bar{b}^k(r_k y, r_k^2 s), \quad \pi^k(y, s) = r_k^2 \bar{\pi}^k(r_k y, r_k^2 s),$$

where $(y, s) \in B_{\frac{1}{r_k}} \times (-\frac{1}{r_k^2}, 0)$. Then it follows from (4.3) that

$$\begin{aligned} A(u^k, b^k; r) + E(u^k, b^k; r) + \tilde{D}(\pi^k, r) &\leq C_1, \quad \text{for all } 0 < r < 1, \\ G(b_h^k, p, q; 1) &\rightarrow 0, \quad \text{as } r_k \rightarrow 0. \\ \|u^k\|_{L_t^\infty L_x^3(B_{\frac{1}{r_k}} \times (-\frac{1}{r_k^2}, 0))} &\leq C_0. \end{aligned} \quad (4.4)$$

For any $a, T > 0$, choose sufficiently large k such that

$$\begin{aligned} \|u^k\|_{L^{\infty,2}((-T,0) \times B_a)} + \|b^k\|_{L^{\infty,2}((-T,0) \times B_a)} \\ + \|\nabla u^k\|_{L^2((-T,0) \times B_a)} + \|\nabla b^k\|_{L^2((-T,0) \times B_a)} &\leq c(a, T). \end{aligned} \quad (4.5)$$

Hence, $u^k \cdot \nabla u^k, u^k \cdot \nabla b^k, b^k \cdot \nabla u^k, b^k \cdot \nabla b^k \in L_t^{\frac{3}{2}} L_x^{\frac{9}{8}}(Q_a)$. This gives by the linear Stokes theory [7] that

$$|\partial_t u^k| + |\Delta u^k| + |\partial_t b^k| + |\Delta b^k| + |\nabla p^k| \in L_t^{\frac{3}{2}} L_x^{\frac{9}{8}}(Q_{3a/4}).$$

Then Lions-Aubin's lemma ensures that there exists (u, b, π) such that for any $a, T > 0$ (up to subsequence),

$$\begin{aligned} u^k &\rightharpoonup u, \quad b^k \rightarrow b, \quad \text{in } L^3((-T, 0) \times B_a), \\ u^k &\rightarrow u, \quad b^k \rightarrow b, \quad \text{in } C([-T, 0]; L^{9/8}(B_a)), \\ \pi^k &\rightharpoonup \pi \quad \text{in } L^{\frac{3}{2}}((-T, 0) \times B_a), \\ \|u\|_{L_t^\infty L_x^3((-T, 0) \times \mathbb{R}^3)} &\leq C_0, \\ b_h^k &\rightharpoonup b_h = 0, \quad \text{in } L^q((-T, 0); L^p(B_a)), \end{aligned}$$

as $k \rightarrow +\infty$.

Hence $\partial_3 b_3 = 0$ and $b \cdot \nabla b = 0$ due to the velocity field equations, and we get

$$u_t - \Delta u + u \cdot \nabla u = -\nabla \pi, \quad \nabla \cdot u = 0. \quad (4.6)$$

Using the property of weak convergence, by (4.4) we have

$$C(u, 1) + \tilde{D}(\pi, 1) \leq C_0,$$

and due to $u \in L_{t,x}^{\infty,3}$, the well-known result in [7] yields that $\|u\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0$. Applying the interior regularity criteria in Proposition 2.3, we obtained that $\|b\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0$.

Since $(0, 0)$ is a singular point of $(v^k, \bar{b}^k, \bar{\pi}^k)$, by small regularity results in Proposition 2.2 we have

$$\varepsilon < C(v^k, r) + C(\bar{b}^k, r) + \tilde{D}(\bar{\pi}^k, r),$$

for any $0 < r < 1$. Thus,

$$\varepsilon < C(u^k, r) + C(b^k, r) + \tilde{D}(\pi^k, r),$$

for any $0 < r < 1$.

Take the supremum limit as $k \rightarrow \infty$, we have

$$\varepsilon < C_0 r^3 + \tilde{D}(\pi^k, r),$$

for any $0 < r < 1$.

By the pressure estimate in Lemma 4.4 and (4.4), we have

$$\tilde{D}(\pi^k, r) \leq C_0 \left(\left(\frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi^k, \rho) + \left(\frac{\rho}{r} \right)^2 C(u^k, b^k; \rho) \right) \leq C_0 \left(\frac{r}{\rho} \right)^{5/2} + C_0 \left(\frac{\rho}{r} \right)^2 C(u^k, b^k; \rho),$$

for any $0 < r < \rho < 1$. Choose $\rho = \sqrt{r}$, then

$$\limsup_{k \rightarrow \infty} \tilde{D}(\pi^k, r) \leq C_0 \sqrt{r},$$

for any $0 < r < 1$.

Hence, we have $\varepsilon < C_0 r^3 + C_0 \sqrt{r}$, for any $0 < r < 1$. Obviously, it's a contradiction. The proof is complete. \square

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